On the Minkowski dimension of Brownian motion with drift

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Abstract

The Cameron Martin theorem states that, for any continuous function f in the Dirichlet space, the law of Brownian motion B and that of $B + f$ are equivalent. In this paper, we consider the Minkowski dimension of $B + f$, when f is not assumed to be in the Dirichlet space.

First, we look at the almost sure constancy of the Minkowski dimension of $B+f$. We then give some lower bounds for the Minkowski dimension of the image of Brownian motion with and without drift, and show that, in dimension 1, the Minkowski dimension of the graph of $B+f$ is equal to the maximum between that of the graph of f and that of the graph of B.

1 Introduction

Let $(B_t, t \leq 1)$ be a Brownian motion and f be a continuous function. By the Cameron-Martin theorem, the law of $B + f$ is equivalent to that of B when f is in the Dirichlet space

$$D[0,1] = \left\{ f \in C[0,1] : f(t) = \int_0^t g(s)ds \text{ for some function } g \in L^2[0,1] \right\},$$

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and singular to the law of B otherwise.

In [PS], Peres and Sousi show that, for f continuous not necessarily in $D[0,1]$, the Hausdorff dimensions of the image and the graph of $B+f$ are almost surely constant. Furthermore, they prove the following lower bounds for those constants.

Theorem 1.1. *Let $(B_t, t \leq 1)$ be a Brownian motion in dimension d, and let $f : [0,1] \to \mathbb{R}^d$ be a continuous function. Then, for every closed subset A of $[0,1]$, we have, almost surely,*

$$\dim(B+f)(A) \geq \max\{(2\dim A) \wedge d, \dim f(A)\}.$$

Theorem 1.2. *Let $(B_t, t \leq 1)$ be a Brownian motion in dimension d, and let $f : [0,1] \to \mathbb{R}^d$ be a continuous function. Then, almost surely,*

$$\dim G_{[0,1]}(B+f) \geq \begin{cases} \max\{3/2, \dim G_{[0,1]}(f)\}, & d = 1, \\ \max\{2, \dim G_{[0,1]}(f)\}, & d \geq 2. \end{cases}$$

Here, we denoted the graph of a function h restricted to A by

$$G_A(h) = \{(t, h(t)) : t \in A\}.$$

In this text, we investigate similar questions for the Minkowski dimension of $B+f$. We will always require that the function f be bounded for definiteness of the Minkowski dimension. However, we will not always need f to be continuous. We will start with the following result on the constancy of the Minkowski dimension of the graph of $B+f$.

Theorem 1.3. *Let $(B_t, t \leq 1)$ be a Brownian motion in dimension d, let $f : [0,1] \to \mathbb{R}^d$ be a bounded measurable function, and let A be a Borel subset of $[0,1]$. Then, there exist constants $\underline{c}$ and $\bar{c}$ such that, almost surely,*

$$\underline{\dim}_M G_A(B+f) = \underline{c}, \quad \text{and} \quad \overline{\dim}_M G_A(B+f) = \bar{c}.$$

Then, we will look at the Minkowski dimension of the image of Brownian motion with a drift. We will show the following results.

Theorem 1.4. *Let $(B_t, t \leq 1)$ be a Brownian motion in dimension d, and let $f : [0,1] \to \mathbb{R}^d$ be a bounded measurable function. Then, for every Borel subset A of $[0,1]$, we have, almost surely,*

$$\underline{\dim}_M (B+f)(A) \geq \underline{\dim}_M f(A),$$
$$\overline{\dim}_M (B+f)(A) \geq \overline{\dim}_M f(A).$$

Theorem 1.5. *Let $(B_t, t \leq 1)$ be a linear Brownian motion, and let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function. Then, for every Borel subset A of $[0,1]$, we have, almost surely,*

$$\underline{\dim}_M (B+f)(A) \geq \frac{2\underline{\dim}_M A}{\underline{\dim}_M A + 1}, \quad \text{and} \quad \overline{\dim}_M (B+f)(A) \geq \frac{2\overline{\dim}_M A}{\overline{\dim}_M A + 1}.$$

In particular, choosing $f = 0$, the inequalities of Theorem 1.5 hold for Brownian motion without a drift. We will show that in that case, the inequalities are sharp in non trivial cases, and attained for the sets

$$A_\beta = \{n^{-\beta} : n \in \mathbb{N}\} \cup \{0\}, \quad \beta \in \mathbb{R}_+. \tag{1.1}$$

Finally, we will look at the Minkowski dimension of the graph of $B + f$. We will prove the following results.

Theorem 1.6. *Let $(B_t, t \leq 1)$ be a Brownian motion in dimension d, and let $f : [0,1] \to \mathbb{R}^d$ be a bounded measurable function. Then, for every Borel subset A of $[0,1]$, we have, almost surely,*

$$\underline{\dim}_M G_A(B+f) \geq \underline{\dim}_M G_A(f)$$
$$\overline{\dim}_M G_A(B+f) \geq \overline{\dim}_M G_A(f).$$

Theorem 1.7. *Let $(B_t, t \leq 1)$ be a linear Brownian motion, let $f : [0,1] \to \mathbb{R}$ be a continuous function. Then, almost surely,*

$$\underline{\dim}_M G_{[0,1]}(B+f) = \max\{3/2, \underline{\dim}_M G_{[0,1]}(f)\},$$
$$\overline{\dim}_M G_{[0,1]}(B+f) = \max\{3/2, \overline{\dim}_M G_{[0,1]}(f)\}.$$

2 Almost sure constancy of the dimension of Brownian motion with a drift

To prove Theorem 1.3, we start with two preliminary results, which show that adding a linear or piecewise affine drift to a function does not alter the Minkowski dimension of its graph.

Proposition 2.1. *Let $f : [0,1] \to \mathbb{R}^d$ be a bounded measurable function, and $\mu \in \mathbb{R}^d$. Define $g : [0,1] \to \mathbb{R}^d$ by*

$$g(t) = f(t) + \mu t.$$

Then, for every Borel subset A of $[0,1]$, we have

$$\underline{\dim}_M G_A(f) = \underline{\dim}_M G_A(g) \quad \textit{and} \quad \overline{\dim}_M G_A(f) = \overline{\dim}_M G_A(g).$$

Proof. For $\varepsilon \in (0,\infty)$ and $k \in \mathbb{N}$, consider $\mathcal{C}_\varepsilon(k)$, the collection of cubes of the form

$$[(k-1)\varepsilon, k\varepsilon] \times \{\text{some cube of edge length } \varepsilon \text{ in } \mathbb{R}^d\},$$

and put

$$\mathcal{C}_\varepsilon = \bigcup_{k\in\mathbb{N}} \mathcal{C}_\varepsilon(k).$$

Write $N = \lceil \|\mu\|_\infty \rceil$, and consider a covering of $G_A(f)$ by cubes of $\mathcal{C}_\varepsilon$. Consider the cubes of the covering that are in $C_\varepsilon(k)$, and thus form a covering of $G_{A\cap[(k-1)\varepsilon,k\varepsilon]}(f)$ by $\mu(k-1)\varepsilon$. Clearly, shifting them by $\mu(k-1)\varepsilon$ produces a covering of

$$G_{A\cap[(k-1)\varepsilon,k\varepsilon]}(f + \mu(k-1)\varepsilon).$$

But within a time interval of length ε, the drift cannot move $f(t)$ by more than $N\varepsilon$ in any given direction. Therefore,

$$G_{A\cap[(k-1)\varepsilon,k\varepsilon]}(g),$$

may be covered with $(2N+1)^d$ as many cubes of $\mathcal{C}_\varepsilon(k)$ as are required to cover

$$G_{A\cap[(k-1)\varepsilon,k\varepsilon]}(f).$$

It follows that the covering number of $G_A(g)$ with elements of $\mathcal{C}_\varepsilon$ is at most $(2N+1)^d$ times that of $G_A(f)$. Therefore,

$$\underline{\dim}_M G_A(g) \le \underline{\dim}_M G_A(f) \quad \text{and} \quad \overline{\dim}_M G_A(g) \le \overline{\dim}_M G_A(f).$$

Since $f(t) = g(t) - \mu t$, the same argument shows that

$$\underline{\dim}_M G_A(f) \le \underline{\dim}_M G_A(g) \quad \text{and} \quad \overline{\dim}_M G_A(f) \le \overline{\dim}_M G_A(g),$$

and completes the proof. □

The invariance of the Minkowski dimension by translation, and splitting the graph into a finite number of subsets immediately proves the following corollary.

Corollary 2.2. *Let $f : [0,1] \to \mathbb{R}^d$ be a bounded measurable function, and $h : [0,1] \to \mathbb{R}^d$ be piecewise affine. Put $g = f + h$. Then, for every Borel subset A of $[0,1]$, we have*

$$\underline{\dim}_M G_A(f) = \underline{\dim}_M G_A(g) \quad \textit{and} \quad \overline{\dim}_M G_A(f) = \overline{\dim}_M G_A(g).$$

We can now proceed with the proof of Theorem 1.3

Proof of Theorem 1.3. We only prove the result for the lower Minkowski dimension. The proof for the upper Minkowski dimension is identical.

Consider Lévy's construction of Brownian motion as

$$B = \lim_{n\to\infty} B_n = \lim_{n\to\infty} \sum_{k=1}^{n} X_k,$$

where $(X_k, k \in \mathbb{N})$ is an independent sequence of continuous piecewise affine random paths on $[0,1]$, and the convergence is uniform on $[0,1]$.

For $n \in \mathbb{N}$, put $Z_n = B - B_{n-1}$. Since, for every n,

$$B + f = Z_n + f + B_{n-1},$$

and B_{n-1} is piecewise affine, Corollary 2.2 implies that

$$\underline{\dim}_M G_A(B + f) = \underline{\dim}_M G_A(Z_n + f).$$

In particular, for any real number a,

$$\{\underline{\dim}_M G_A(B + f) \le a\} = \{\underline{\dim}_M G_A(Z_n + f) \le a\} \in \sigma(X_k, k \ge n).$$

Since this is true for every n, the event

$$\{\underline{\dim}_M G_A(B + f) \le a\}$$

is in the tail σ-algebra. Therefore, by Kolmogorov's 0-1 law, it has probability 0 or 1. It follows that the Minkowski dimension of $G_A(B+f)$ is almost surely constant, as required. $\square$

3 Minkowski dimension of the image of Brownian motion with a drift

We start this section with a technical estimate provided in the following lemma, which we will need to prove Theorem 1.4.

Lemma 3.1. *Let $(B_t, t \leq 1)$ be a Brownian motion in dimension d, let $f : [0,1] \to \mathbb{R}^d$ be a bounded measurable function, and let A be a Borel subset of $[0,1]$. Let $\varepsilon \in (0,\infty)$, and consider a packing of $f(A)$ with P balls of radius ε*

$$\{\mathcal{B}(f(t_i),\varepsilon) : t_i \in A \text{ and } i \in \{1,\dots,P\}\}.$$

For every $i \in \{1,\dots,P\}$, put

$$N_i = \#\{j \neq i : |(B+f)(t_i) - (B+f)(t_j)| < \varepsilon\}.$$

Then there exist positive constants c and γ such that,

$$\mathbf{E}N_i \leq c\log(1/\varepsilon)^{d+1} + O(\varepsilon^\gamma), \quad \text{for all } i \in \{1,\dots,P\},$$

as $\varepsilon \to 0$.

In the proof, we will let the values of c and γ change from line to line.

Proof. Let us consider only $\varepsilon \in (0,\varepsilon_0)$ for some $\varepsilon_0 \in (0,1)$.

For $k \in \mathbb{N}$, define

$$\begin{aligned}
S_i(k) &= \{j : |f(t_i) - f(t_j)| \in [2^k\varepsilon, 2^{k+1}\varepsilon)\},\\
S_i'(k) &= \left\{ j \in S_i(k) : |t_i - t_j| \geq \left(\frac{2^k\varepsilon}{\log(1/\varepsilon)}\right)^2 \right\},\\
S_i''(k) &= S_i(k) \setminus S_i'(k).
\end{aligned}$$

Since f is bounded, say by M, we see that $S_i(k) = \varnothing$ when $2^k\varepsilon > 2M$, or, equivalently, when

$$k > \log_2(2M) + \log_2(1/\varepsilon) = n(\varepsilon),$$

say.

Notice also that the balls $\mathcal{B}(f(t_j), \varepsilon)$, with $j \in S_i(k)$, are disjoint and included in $\mathcal{B}(f(t_i), (2^{k+1}+1)\varepsilon)$. Therefore, there are at most

$$\frac{\mathrm{vol}(\mathcal{B}(0, (2^{k+1}+1)\varepsilon))}{\mathrm{vol}(\mathcal{B}(0, \varepsilon))} \le c 2^{dk+d} \tag{3.1}$$

of them.

By definition,

$$\mathbf{E}N_i = \sum_{\substack{j=1 \\ j \neq i}}^{P} p(i,j) = \sum_{k=1}^{n(\varepsilon)} \sum_{j \in S_i'(k)} p(i,j) + \sum_{k=1}^{n(\varepsilon)} \sum_{j \in S_i''(k)} p(i,j),$$

where we used the shorthand notation

$$p(i,j) = \mathbf{P}(|(B+f)(t_i) - (B+f)(t_j)| < \varepsilon).$$

On the one hand, when $j \in S_i'(k)$, we have

$$\begin{aligned} p(i,j) &= \frac{1}{(2\pi|t_i - t_j|)^{d/2}} \int_{\mathcal{B}(f(t_i)-f(t_j), 2\varepsilon)} \exp\left\{ -\frac{|x|^2}{2|t_i - t_j|} \right\} dx \\ &\le c \frac{\log(1/\varepsilon)^d}{2^{dk}\varepsilon^d} \mathrm{vol}(\mathcal{B}(0, 2\varepsilon)) \\ &= c\, 2^{-dk} \log(1/\varepsilon)^d. \end{aligned} \tag{3.2}$$

And on the other hand, when $j \in S_i''(k)$, we have

$$\begin{aligned} p(i,j) &\le \mathbf{P}(|B(t_i) - B(t_j)| > |f(t_i) - f(t_j)| - \varepsilon) \\ &\le \mathbf{P}(|B(t_i) - B(t_j)| > \alpha_k \varepsilon) \\ &\le c \frac{|t_i - t_j|^{1/2}}{\tilde{c}\alpha_k \varepsilon} \exp\left\{ -\frac{\tilde{c}^2 \alpha_k^2 \varepsilon^2}{2|t_i - t_j| d^2} \right\} \\ &\le c \frac{2^k}{\alpha_k} \frac{\varepsilon^{\gamma \log(1/\varepsilon_0)\alpha_k^2/2^{2k}}}{\log(1/\varepsilon)}, \end{aligned} \tag{3.3}$$

where we put $\alpha_k = 2^k - 1$, and $\tilde{c}$ is a positive constant whose existence is guaranteed by a tail estimate similar to that of Lemma 12.9 of [MP10].

Now notice that, since $\varepsilon < \varepsilon_0$, we have

$$n(\varepsilon) \le c \log(1/\varepsilon),$$

and that the ratio $\alpha_k/2^k$ is bounded. Together with (3.1), (3.2) and (3.3), this shows that

$$\sum_{k=1}^{n(\varepsilon)} \sum_{j \in S_i'(k)} p(i,j) \leq c \log(1/\varepsilon)^{d+1}, \quad \text{and} \quad \sum_{k=1}^{n(\varepsilon)} \sum_{j \in S_i''(k)} p(i,j) \leq c\varepsilon^{\gamma},$$

assuming that ε_0 is small enough. Summing these two terms completes the proof. $\square$

We may now give the proof of Theorem 1.4. Henceforth, we will use the letter c to mean *some constant* whose value may change from an expression to the next.

Proof of Theorem 1.4. To simplify the notation, write

$$\alpha = \underline{\dim}_M f(A), \quad \text{and} \quad \beta = \overline{\dim}_M f(A).$$

Consider a packing of $f(A)$ with $P_\varepsilon(f)$ balls of radius ε

$$\{\mathcal{B}(f(t_i), \varepsilon) : t_i \in A \text{ and } i \in \{1, \ldots, P_\varepsilon(f)\}\}.$$

For $C \in (0, \infty)$, call a point $f(t_i)$ *good* if

$$N_i = \#\{j \neq i : |(B+f)(t_i) - (B+f)(t_j)| < \varepsilon\} \leq C\mathbf{E}N_i,$$

and *bad* otherwise.

We have

$$\mathbf{P}(\text{number of good points} \geq P_\varepsilon(f)/2) \geq 1 - 2/C,$$

because, by Markov's inequality, the probability of having at least $P_\varepsilon(f)/2$ bad points is at most

$$\mathbf{P}\left(\sum_{i=1}^{P_\varepsilon(f)} \mathbf{1}_{N_i > C\mathbf{E}N_i} \geq \frac{P_\varepsilon(f)}{2}\right) \leq \frac{2}{P_\varepsilon(f)} \sum_{i=1}^{P_\varepsilon(f)} \mathbf{P}(N_i \geq C\mathbf{E}N_i) \leq \frac{2}{C}.$$

For every good t_i, remove the N_i balls $\mathcal{B}((B+f)(t_j), \varepsilon)$ that intersect $\mathcal{B}((B+f)(t_i), \varepsilon)$, and remove all the balls centered at bad points. When the number of good points is at least $P_\varepsilon(f)/2$, this leaves us with a packing of $(B+f)(A)$ of at least

$$\frac{P_\varepsilon(f)/2}{1 + C \max_i \mathbf{E} N_i}$$

balls of radius $\varepsilon/2$. Then, the estimate of Lemma 3.1 shows that, for ε small enough, this is at least

$$\frac{P_\varepsilon(f)/2}{1 + C(c \log(1/\varepsilon)^{d+1} + O(\varepsilon^\gamma))} \geq c \frac{P_\varepsilon(f)}{C \log(1/\varepsilon)^{d+1}}. \tag{3.4}$$

Let us first look at the upper Minkowski dimension. Let $(\varepsilon_n, n \in \mathbb{N})$ be a sequence of positive real numbers converging to 0 along which

$$\frac{\log P_{\varepsilon_n}(f)}{\log(1/\varepsilon_n)} \to \beta.$$

Fix $\zeta \in (0, \infty)$. For all n large enough, we have

$$P_{\varepsilon_n}(f) \geq \varepsilon_n^{-\beta+\zeta}.$$

Together with (3.4), this yields that, for n large enough,

$$\mathbf{P}\left(\frac{\log P_{\varepsilon_n/2}(B+f)}{\log(1/\varepsilon_n)} \geq \beta - \zeta + r(C, \varepsilon_n)\right) \geq 1 - \frac{2}{C}, \tag{3.5}$$

where

$$r(C, \varepsilon) = \frac{1}{\log(1/\varepsilon)} \log\left(\frac{c}{C \log(1/\varepsilon)^{d+1}}\right) \to 0,$$

as $\varepsilon \to 0$.

The reverse Fatou lemma then shows that

$$\begin{aligned}
&\mathbf{P}\left(\limsup_{n\to\infty} \frac{\log P_{\varepsilon_n/2}(B+f)}{\log(1/\varepsilon_n)} \geq \beta - \zeta\right) \\
&\geq \limsup_{n\to\infty} \mathbf{P}\left(\frac{\log P_{\varepsilon_n/2}(B+f)}{\log(1/\varepsilon_n)} \geq \beta - \zeta + r(C, \varepsilon_n)\right) \\
&\geq 1 - \frac{2}{C}.
\end{aligned}$$

Since C and ζ are arbitrary, it follows that, almost surely,

$$\overline{\dim}_M (B+f)(A) \geq \beta.$$

Let us now look at the lower Minkowski dimension. We will use that it is sufficient to look at packing with dyadic cubes. Put $\tilde{\varepsilon}_n = 2^{-n}$, and notice that $|r(n^2, \tilde{\varepsilon}_n)| \to 0$ as $n \to \infty$. For n large enough that the estimate (3.4) holds, we have

$$\mathbf{P}\left(\frac{\log P_{\tilde{\varepsilon}_n/2}(B+f)}{\log(1/\tilde{\varepsilon}_n)} - r(n^2, \tilde{\varepsilon}_n) \leq \frac{\log P_{\tilde{\varepsilon}_n/2}(f)}{\log(1/\tilde{\varepsilon}_n)} \right) \leq \frac{2}{n^2}.$$

So, the Borel-Cantelli lemma implies that

$$\mathbf{P}\left(\frac{\log P_{\tilde{\varepsilon}_n}(B+f)}{\log(1/\tilde{\varepsilon}_n)} - r(n^2, \tilde{\varepsilon}_n) \geq \frac{\log P_{\tilde{\varepsilon}_n}(f)}{\log(1/\tilde{\varepsilon}_n)} \text{ eventually} \right) = 1.$$

Since, almost surely,

$$\frac{\log P_{\tilde{\varepsilon}_n}(f)}{\log(1/\tilde{\varepsilon}_n)} \geq \alpha - \zeta, \text{ eventually},$$

$r(n^2, \tilde{\varepsilon}_n) \to 0$ as $n \to \infty$ and ζ is arbitrary, we conclude that, almost surely,

$$\underline{\dim}_M (B+f)(A) \geq \alpha,$$

which completes the proof. $\square$

We now turn to the case where $d = 1$, and give the proof of Theorem 1.5.

Proof of Theorem 1.5. To simplify the notation, put

$$\alpha = \underline{\dim}_M A, \quad \text{and} \quad \beta = \overline{\dim}_M A.$$

Consider a packing of A with $P_\delta(A)$ balls of radius δ

$$\{\mathcal{B}(t_i, \delta) : t_i \in A \text{ and } i \in \{1, \ldots, P_\delta(A)\}\}.$$

For ε and $C \in (0, \infty)$, call a point t_i *good* if

$$N_i = \#\{j \neq i : |(B+f)(t_i) - (B+f)(t_j)| < 2\varepsilon\} \leq C\mathbf{E}N_i,$$

and *bad* otherwise.

We have

$$\mathbf{P}(\text{number of good points} \geq P_\delta(A)/2) \geq 1 - 2/C,$$

because, by Markov's inequality, the probability of having at least $P_\delta(A)/2$ bad points is

$$\mathbf{P}\left(\sum_{i=1}^{P_\delta(A)} \mathbf{1}_{N_i > C\mathbf{E}N_i} \geq \frac{P_\delta(A)}{2}\right) \leq \frac{2}{P_\delta(A)} \sum_{i=1}^{P_\delta(A)} \mathbf{P}(N_i \geq C\mathbf{E}N_i) \leq \frac{2}{C}.$$

For every good t_i, remove the N_i balls $\mathcal{B}(t_j,\delta)$ that intersect $\mathcal{B}(t_i,\delta)$, and remove all the balls centered at bad points. When the number of good points is at least $P_\delta(A)/2$, this leaves us with a packing of $B(A)$ of at least

$$\frac{P_\delta(A)/2}{1 + C\max_i \mathbf{E}N_i} \tag{3.6}$$

balls with radius ε.

Now, using the properties of the Gaussian distribution and that $|t_{i+\ell} - t_i| \geq \ell\delta$ (assume that $t_1 \leq \cdots \leq t_{P_\delta(A)}$), we can bound $\mathbf{E}N_i$ by

$$\begin{aligned}\mathbf{E}N_i &= \sum_{\substack{j=1\\ j\neq i}}^{P_\delta(A)} \mathbf{P}(|(B+f)(t_i) - (B+f)(t_j)| < 2\varepsilon)\\ &\leq c \sum_{\substack{\ell=-i+1\\ \ell\neq 0}}^{P_\delta(A)-i} \frac{\varepsilon}{\sqrt{|\ell|\delta}}\\ &\leq c\frac{\varepsilon}{\sqrt{\delta}}\sqrt{P_\delta(A)}.\end{aligned} \tag{3.7}$$

Let us first look at the upper Minkowski dimension. Let $(\delta_n, n \in \mathbb{N})$ be a sequence of positive numbers converging to 0 along which

$$\beta_n = \frac{\log P_{\delta_n}(A)}{\log(1/\delta_n)} \to \beta.$$

For $n \in \mathbb{N}$, put $\varepsilon_n = \delta_n^{(\beta_n+1)/2}$. For this choice, (3.7) becomes

$$\mathbf{E}N_i \leq c\varepsilon_n\delta_n^{-(\beta_n+1)/2} = c.$$

Therefore, the expression in (3.6) is at least

$$\frac{P_{\delta_n}(A)/2}{1+Cc} \geq \frac{c}{C}P_{\delta_n}(A) = \frac{c}{C}\delta_n^{-\beta_n} = \frac{c}{C}\varepsilon_n^{-2\beta_n/(\beta_n+1)},$$

for C large enough. This yields

$$\mathbf{P}\left(\frac{\log P_{\varepsilon_n}((B+f)(A))}{\log(1/\varepsilon_n)} \geq \frac{2\beta_n}{\beta_n+1} + r(C,\varepsilon_n)\right) \geq 1-2/C,$$

where

$$r(C,\varepsilon) = \frac{\log(c/C)}{\log(1/\varepsilon)} \to 0,$$

as $\varepsilon \to 0$.

The reverse Fatou lemma then shows that

$$\begin{aligned} &\mathbf{P}\left(\limsup_{n\to\infty} \frac{\log P_{\varepsilon_n}((B+f)(A))}{\log(1/\varepsilon_n)} \geq \frac{2\beta}{\beta+1}\right) \\ &\geq \limsup_{n\to\infty} \mathbf{P}\left(\frac{\log P_{\varepsilon_n}((B+f)(A))}{\log(1/\varepsilon_n)} \geq \frac{2\beta}{\beta+1} + r(C,\varepsilon_n)\right) \\ &\geq 1 - \frac{2}{C}. \end{aligned} \tag{3.8}$$

Since C is arbitrary, it follows that, almost surely,

$$\overline{\dim}_M((B+f)(A)) \geq \frac{2\beta}{\beta+1}.$$

Let us now look at the lower Minkowski dimension. Put $\tilde{\varepsilon}_n = 2^{-n}$, and notice that $|r(n^2,\tilde{\varepsilon}_n)| \to 0$ as $n \to \infty$.

For $n \in \mathbb{N}$, define $\tilde{\delta}_n$ and α_n by

$$\tilde{\delta}_n = \tilde{\varepsilon}_n^{2/(\alpha+1)}, \quad \text{and} \quad P_{\tilde{\delta}_n}(A) = \tilde{\delta}_n^{-\alpha_n}.$$

For $\zeta \in (0,\infty)$, we have $\alpha_n \geq \alpha - \zeta$ eventually since

$$\liminf_{n\to\infty} \alpha_n \geq \alpha.$$

Also, for these choices, (3.7) becomes

$$\mathbf{E}N_i \leq c\tilde{\varepsilon}_n \tilde{\delta}_n^{-(\alpha_n+1)/2} = c\tilde{\varepsilon}_n^{(\alpha-\alpha_n)/(\alpha+1)}. \tag{3.9}$$

Then, the expression in (3.6) is at least

$$\frac{P_{\tilde{\delta}_n}(A)/2}{1+cC\tilde{\varepsilon}_n^{(\alpha-\alpha_n)/(\alpha+1)}} \geq \frac{c}{C}\frac{\tilde{\varepsilon}_n^{-2\alpha_n/(\alpha+1)}}{\tilde{\varepsilon}_n^{(\alpha-\alpha_n-2\zeta)/(\alpha+1)}} \geq \frac{c}{C}\tilde{\varepsilon}_n^{-(\alpha_n+\alpha-2\zeta)/(\alpha+1)},$$

for C large enough. Reasoning as in (3.8) then gives

$$\mathbf{P}\left(\frac{\log P_{\tilde{\varepsilon}_n}((B+f)(A))}{\log(1/\tilde{\varepsilon}_n)} \leq \frac{\alpha_n + \alpha - 2\zeta}{\alpha+1} + r(n^2, \tilde{\varepsilon}_n)\right) \leq \frac{2}{n^2}.$$

Now, the Borel and Cantelli lemma implies that

$$\begin{aligned}
&\mathbf{P}\left(\frac{\log P_{\tilde{\varepsilon}_n}((B+f)(A))}{\log(1/\tilde{\varepsilon}_n)} - r(n^2, \tilde{\varepsilon}_n) \geq \frac{\alpha_n + \alpha - 2\zeta}{\alpha+1} \text{ eventually}\right) \\
&\geq \mathbf{P}\left(\frac{\log P_{\tilde{\varepsilon}_n}((B+f)(A))}{\log(1/\tilde{\varepsilon}_n)} - r(n^2, \tilde{\varepsilon}_n) \geq \frac{2\alpha - 3\zeta}{\alpha+1} \text{ eventually}\right) \\
&= 1.
\end{aligned}$$

Since $r(n^2, \tilde{\varepsilon}_n) \to 0$ as $n \to \infty$ and ζ is arbitrary, we conclude that, almost surely,

$$\underline{\dim}_M (B+f)(A) \geq \frac{2\alpha}{\alpha+1},$$

which completes the proof. $\square$

We now show that when $f = 0$, these inequalities are attained for the set A_β defined in (1.1). We start with the following proposition.

Proposition 3.2. *Let $g : \mathbb{R}_+ \to \mathbb{R}$ be a γ-Hölder continuous function and $\beta \in (0, \infty)$. Then*

$$\overline{\dim}_M g(A_\beta) \leq \frac{1}{1+\gamma\beta}.$$

Proof. Since the Minkowski dimension is translation invariant, we may and will assume that $g(0) = 0$. Let C be the Hölder constant of g. For $k \in \mathbb{N}$ and $n \geq k$, we have

$$g(n^{-\beta}) \leq C n^{-\gamma\beta} \leq C k^{-\gamma\beta}.$$

Fix $\varepsilon > 0$. The set $\{n^{-\beta} : n > k\} \cup \{0\}$ may be covered with $Ck^{-\gamma\beta}/\varepsilon$ closed balls of diameter ε; and the set $\{n^{-\beta} : n \leq k\}$ may be covered with k such closed balls. Therefore, the covering number satisfies

$$N(\varepsilon) \leq C\frac{k^{-\gamma\beta}}{\varepsilon} + k.$$

Optimizing over k shows that

$$N(\varepsilon) \leq c\varepsilon^{-1/(1+\gamma\beta)},$$

and the result follows immediately. $\square$

We may now conclude that the inequality in Theorem 1.5 is sharp.

Corollary 3.3. *For every $\alpha \in [0,1]$, there exists a subset A of $\mathbb{R}_+$ of Minkowski dimension α such that, almost surely,*

$$\dim_M B(A) = \frac{2\dim_M A}{1+\dim_M A}.$$

Proof. The case $\alpha = 0$ is clear. So let us assume that $\alpha > 0$ and put

$$\beta = 1/\alpha - 1.$$

It is easy to check that

$$\dim_M A_\beta = \frac{1}{1+\beta} = \alpha.$$

Now pick $\gamma < 1/2$. Since almost all Brownian paths are γ-Hölder continuous, Proposition 3.2 guarantees that

$$\overline{\dim}_M B(A_\beta) \le \frac{1}{1+\gamma\beta} \to \frac{1}{1+\beta/2} = \frac{2\alpha}{\alpha+1},$$

as we let $\gamma \to 1/2$. Applying Theorem 1.5 concludes the proof. $\square$

4 Minkowski dimension of the graph of Brownian motion with a drift

We now look at the dimension of the graph of Brownian motion with a drift. We start with proof of Theorem 1.6.

Proof of Theorem 1.6. The proof is a mixture of the arguments used to prove Theorems 1.6 and 1.5. Therefore, we will essentially highlight the differences without dwelling on the details. We start with an estimate similar to that provided in Lemma 3.1.

Consider a packing of $G_A(f)$ with $\hat{P}_\delta(f)$ balls of radius δ

$$\big\{\mathcal{B}\big((t_i, f(t_i)), \delta\big) : t_i \in A \text{ and } i \in \{1, \dots, \hat{P}_\delta(f)\}\big\}.$$

For ε and $C > 0$, call a point $(t_i, f(t_i))$ *good* if

$$\hat{N}_i = \#\{j \neq i : |(t_i, (B+f)(t_i)) - (t_j, (B+f)(t_j))| < 2\varepsilon\} \le C\mathbf{E}\hat{N}_i,$$

and *bad* otherwise. For $k \in \mathbb{N}$, put

$$\begin{aligned}
\hat{S}_i(k) &= \{j : |(t_i, f(t_i)) - (t_j, f(t_j))| \in [2^k\delta, 2^{k+1}\delta)\},\\
\hat{S}_i'(k) &= \left\{ j \in \hat{S}_i(k) : |t_i - t_j| \geq \left(\frac{2^k\delta}{\log(1/\delta)}\right)^2 \right\},\\
\hat{S}_i''(k) &= \hat{S}_i(k) \setminus \hat{S}_i'(k).
\end{aligned}$$

Since A and f are bounded, we see that $\hat{S}_i(k) = \varnothing$ whenever

$$k > c + \log_2(1/\delta) = \hat{n}(\delta),$$

say. Furthermore, comparing volumes shows that there are at most

$$\frac{\mathrm{vol}(\mathcal{B}(0, (2^{k+1}+1)\delta))}{\mathrm{vol}(\mathcal{B}(0,\delta))} \leq c2^{(d+1)k}$$

elements in $\hat{S}_i(k)$.

Put

$$\begin{aligned}
q(i,j) &= \mathbf{P}\Big(\big|\big(t_i, (B+f)(t_i)\big) - \big(t_j, (B+f)(t_j)\big)\big| < 2\varepsilon\Big)\\
&\leq \mathbf{P}(|(B+f)(t_i) - (B+f)(t_j)| < 2\varepsilon).
\end{aligned}$$

Proceeding as for the estimate for $p(i,j)$ in Lemma 3.1 gives

$$q(i,j) \leq c2^{-dk}\log(1/\delta)^d\varepsilon^d\delta^{-d}.$$

Now fix $\zeta \in (1,\infty)$. Clearly, for all δ small enough,

$$\hat{n}(\delta) \leq \zeta \log_2(1/\delta).$$

Therefore, we get that

$$\begin{aligned}
\sum_{k=1}^{\hat{n}(\delta)} \sum_{j \in \hat{S}_i'(k)} q(i,j) &\leq c\sum_{k=1}^{\hat{n}(\delta)} 2^k \log(1/\delta)^d\varepsilon^d\delta^{-d}\\
&= c\hat{n}(\delta)2^{\hat{n}(\delta)}\log(1/\delta)^d\varepsilon^d\delta^{-d}\\
&= c\delta^{-\zeta}\log(1/\delta)^{d+1}\varepsilon^d\delta^{-d},
\end{aligned}$$

for δ small enough.

On the other hand, provided that $\delta \geq \varepsilon$, reasoning as in (3.3), we get

$$\sum_{k=1}^{\hat{n}(\delta)} \sum_{j \in \hat{S}_i''(k)} q(i,j) = \sum_{k=1}^{\hat{n}(\delta)} \sum_{j \in \hat{S}_i''(k)} \mathbf{P}(|B(t_i) - B(t_j)| > |f(t_i) - f(t_j)| - \delta) \leq c\delta^{\gamma},$$

for some $\gamma \in (0, \infty)$, when δ is small enough.

Let $\theta \in (0, 1/\zeta)$, and consider the choice $\delta = \varepsilon^{\theta} \geq \varepsilon$.

Let us first look at the upper Minkowksi dimension. Let $(\delta_n, n \in \mathbb{N})$ be a sequence of positive numbers converging to 0 along which

$$\beta_n = \frac{\log \hat{P}_{\delta_n}(f)}{\log(1/\delta_n)} \to \beta = \overline{\dim}_M G_A(f).$$

Using Markov's inequality, we see that, with probability at least $1 - 2/C$, we get a packing of $G_A(B + f)$ with at least

$$\frac{\hat{P}_{\delta_n}(f)/2}{1C + C\delta_n^{-\zeta} \log(1/\delta_n)^{d+1}} \geq \frac{c\delta_n^{-\beta}}{C\delta_n^{-\zeta} \log(1/\delta_n)^{d+1}}$$

balls of radius ε; here we used that $\varepsilon/\delta \leq 1$. In other words

$$\mathbf{P}\left(\frac{\log(\hat{P}_{\varepsilon_n}(B + f))}{\log(1/\varepsilon_n)} \geq \theta\beta_n + r(C, \varepsilon_n)\right) \geq 1 - \frac{2}{C},$$

where

$$r(C, \varepsilon) = \frac{1}{\log(1/\varepsilon_n)} \log\left(\frac{c}{C\varepsilon_n^{-\theta\zeta} \log(1/\varepsilon_n^{\theta})^{d+1}}\right) \to 0,$$

since $\theta\zeta < 1$. Using the reverse Fatou lemma, and that C is arbitrary, we see that, almost surely,

$$\overline{\dim}_M G_A(B + f) \geq \theta\beta.$$

Since θ can be made arbitrarily close to 1, we have the result.

To prove the result for the lower Minkowski dimension, it suffices to use the Borel-Cantelli lemma as in the proof of Theorem 1.6. □

We now look at the case where $d = 1$. We start with a general result on the Minkowski dimensions of the graph of the sum of two continuous functions.

Proposition 4.1. *Let f and $g : [0,1] \to \mathbb{R}$ be two continuous functions, and let A be a measurable subset of $[0,1]$. Assume that $\dim_M G_{[0,1]}(f)$ exists. Then,*

$$\underline{\dim}_M G_{[0,1]}(f+g) \le \max\{\dim_M G_{[0,1]}(f), \underline{\dim}_M G_{[0,1]}(g)\},$$
$$\overline{\dim}_M G_{[0,1]}(f+g) \le \max\{\dim_M G_{[0,1]}(f), \overline{\dim}_M G_{[0,1]}(g)\}.$$

Furthermore, in both cases, when the dimensions on the right hand side are different, we even have equality.

Proof. We shall only do the proof for the lower Minkowski dimension when $\dim_M G_{[0,1]}(f) \le \underline{\dim}_M G_{[0,1]}(g)$. The other cases are proved in a similar fashion. To simplify the notation, write

$$\alpha = \dim_M G_{[0,1]}(f), \quad \beta = \underline{\dim}_M G_{[0,1]}(g), \quad \text{and} \quad \gamma = \underline{\dim}_M G_{[0,1]}(f+g).$$

Consider the collection of squares

$$\mathcal{C}_\varepsilon = \{[(k-1)\varepsilon, k\varepsilon] \times [(\ell-1)\varepsilon, \ell\varepsilon] : k \text{ and } \ell \in \mathbb{Z}\}.$$

A covering of $G_{[0,1]}(h)$ is given by taking all the elements of $\mathcal{C}_\varepsilon$ that are hit by it; and that many are needed. Let $S_\varepsilon(h)$ be the number of these squares. We have

$$S_\varepsilon(h) = \frac{1}{\varepsilon} \sum_{k=1}^{1/\varepsilon} \max_{s,t \in [(k-1)\varepsilon, k\varepsilon]} |h(t) - h(s)| = \frac{1}{\varepsilon} \sum_{k=1}^{1/\varepsilon} \Omega_k^\varepsilon(h),$$

say. By considering only the squares of the $\mathcal{C}_\varepsilon$ of the covering corresponding to k and ℓ either even and even, or even and odd, or odd and even, or odd and odd, we may preserve at least $S_h(\varepsilon)/4$ squares. They can then be centered on points of $G_{[0,1]}(h)$ to create a packing. Therefore, the packing number $P_\varepsilon(h)$ and covering number $N_\varepsilon(h)$ of $G_{[0,1]}(h)$ satisfy

$$cS_\varepsilon(h) \le P_\varepsilon(h) \le N_\varepsilon(h) \le S_\varepsilon(h). \tag{4.1}$$

Notice that

$$\Omega_k^\varepsilon(g) - \Omega_k^\varepsilon(f) \le \Omega_k^\varepsilon(f+g) \le \Omega_k^\varepsilon(g) + \Omega_k^\varepsilon(f),$$

and therefore, by (4.1),

$$cN_\varepsilon(g) - N_\varepsilon(f) \le N_\varepsilon(f+g) \le \frac{1}{c} N_\varepsilon(g) + \frac{1}{c} N_\varepsilon(f). \tag{4.2}$$

Let $(\varepsilon_n, n \in \mathbb{N})$ be a sequence of positive real numbers converging to 0 along which

$$\frac{\log N_{\varepsilon_n}(g)}{\log(1/\varepsilon_n)} \to \beta,$$

and fix $\delta \in (0,\infty)$. For all n large enough, we have $N_{\varepsilon_n}(f) \leq \varepsilon_n^{-\alpha-\delta}$, $N_{\varepsilon_n}(g) \leq \varepsilon_n^{-\beta-\delta}$ and $N_{\varepsilon_n}(f+g) \geq \varepsilon_n^{-\gamma+\delta}$. Together with (4.2), this shows that, for all n large enough,

$$\varepsilon_n^{-\gamma+\delta} \leq \frac{1}{c}\varepsilon_n^{-\alpha-\delta} + \frac{1}{c}\varepsilon_n^{-\beta-\delta} \leq C\varepsilon_n^{-\beta-\delta}.$$

Since δ is arbitrary, this gives $\gamma \leq \beta$, as required.

When $\alpha < \beta$, a similar reasoning using (4.2) shows that $\gamma \geq \beta$, and completes the proof. □

We may now give the proof of Theorem 1.7.

Proof of Theorem 1.7. We shall only do the proof for the lower Minkowski dimension. The other case is proved the same way.

Suppose first that $\underline{\dim}_M G_{[0,1]}(f) \neq 3/2$. Then, since the Minkwoski dimension of the graph of Brownian motion exists and is $3/2$ almost surely, the result follows from Proposition 4.1.

When $\underline{\dim}_M G_{[0,1]}(f) = 3/2$, Theorem 1.2 ensures that, almost surely,

$$\underline{\dim}_M G_{[0,1]}(B+f) \geq \dim_M G_{[0,1]}(B+f) \geq 3/2,$$

while Proposition 4.1 ensures that, almost surely,

$$\overline{\dim}_M G_{[0,1]}(B+f) \leq 3/2.$$ □

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